Correction Factors for Mindlin Higher-order Plate Theory with the Consideration of Electrodes

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Abstract—**Mindlin plate theory has been the choice for the analysis of high frequency vibrations of piezoelectric quartz crystal resonators, which utilize electroded crystal plates vibrating at the thickness-shear and overtone modes to achieve frequency generation functions and applications. For this purpose, Mindlin plate equations have been simplified, modified, and corrected to accommodate the actual geometry, structural complications, practical boundary conditions, and various bias fields such as temperature change and initial stresses. Accordingly, the correction of Mindlin plate theory should also include complications through factors which adjust the thickness-shear frequencies to the exact three-dimensional solutions from earlier studies with the first-order plate equations. In recent studies, correction factors for equations up to the third-order theory have been introduced and they have the capability to make the cut-off frequencies for the thickness-shear modes accurate up to much higher-orders and improving the accuracy of the extensional mode groups also. These correction factors are derived based on the general principle of comparison and matching of exact frequencies of interested vibration modes in the coupled two-dimensional equations, and the inclusion of structural complications such as electrodes is an natural extension of the derivation with practical importance. In this study, we consider the effect of thin electrodes through their mass loading formulation by adding the mass ratio in the inertia terms as demonstrated by Mindlin and others. Consequently, a set of frequency equations for both thickness-shear and extensional modes are established with the inclusion of mass ratio of electrodes. These equations are solved numerically for mass ratios to obtain the corresponding correction factors which have been modified by the presence of electrodes. These correction factors can be utilized in the improvement of applications of the higher-order plate equations with the finite element implementation, which remains as one of the reliable tool in the precise analysis and design of quartz crystal resonators. The derivation of these correction factors can be further incorporated into nonlinear equations of piezoelectric solids.**

I. INTRODUCTION

Mindlin plate theory based on the power series expansion of displacements has been widely used in the analysis and design of quartz crystal resonators through analytical solutions and numerical solutions based on the finite element implementation with approximations in between. Extensive research efforts have been made on Mindlin plate theory in the filed of high frequency vibrations of piezoelectric plates, as Mindlin developed the systematic higher-order equations for the analysis of crystal resonators specifically. In addition to the well-known and highly cited paper on the Mindlin plate equation [1], a more systematic introduction was given by Mindlin in 1972 [2], and the complete monograph on Mindlin plate theory is recently published by World Scientific [3], while the original version by the US Army Signal Corps is available through the Internet [4]. The applications of Mindlin plate theory has been extended to various plate structures with composite in particular and modifications have been made to satisfy special requirements.

For piezoelectric resonator applications, Mindlin plate theory is still the most convenient and effective tool for the analysis in design improvements and applications. As a result, there are many efforts on the simplification, correction, and truncation. The main objective of such efforts is to simplify the equations for the coupled thickness-shear vibrations of piezoelectric crystal plates with complications such as electrodes [5-7], thermal effect [8], and correction [9]. In addition to analytical solutions based straight-crested wave assumptions [10], one important application is the finite element method with Mindlin plate theory for the analysis of quartz crystal resonators [11-13]. Since the finite element implementation usually include all higher-order displacements below a chosen order rather than the selected modes in analytical solutions, the influence of higher-order displacements are particularly of interests. Actually, we have found that the accuracy in the frequency and mode shapes of certain modes requires higher-order plate equations [11], and the correction related to these modes is certainly important also. As a result, Mindlin's correction scheme of making cut-off frequencies exact for higher-order modes are extended to the third-order or even higher modes with new sets of correction factors [9], and exact frequencies for the fundamental thickness-shear mode with the consideration of piezoelectric effect and electrodes are also obtained [14, 15]. These results are now ready to be combined for the complete correction of Mindlin plate equations of higher-order modes

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with the consideration of electrode effect also. The corrections will refine current finite element programs to meet increasing requirement on accuracy as the devices shrinking and effects of structural complications being amplified.

II. CORRECTION OF MINDLIN PLATE EQUATIONS

Although the Mindlin plate theory is obtained through a systematic expansion of the mechanical displacements and electrical potential in the thickness coordinate, the accuracy of the plate theory cannot be improved by simply increasing the number of vibration modes, or the order of plate equations [9]. As a remedy, correction factors have been introduced to make the truncated equations with limited number of terms more accurate [2]. The effective procedure was proposed by Mindlin for the first-order equations [3, 5], and we extended it to higher-order equations in a systematic manner [9]. In this paper, we also include the effect of thin electrodes, which are essential in quartz crystal resonators, in the way demonstrated by our earlier study [9].

A. Mindlin Higher-order Plate Equations

The basic equations of Mindlin plate theory are [2, 9]

$$
\begin{aligned}
& u_j = \sum_{n=0} u_j^{(n)} x_2^n, \phi = \sum_{n=0} \phi^{(n)} x_2^{(n)}, \ j = 1,2,3; \\
& \overline{K} = \frac{1}{2}\rho \sum_m \sum_n B_{mn} \dot{u}_j^{(m)} \dot{u}_j^{(n)}, \\
& \overline{H} = \frac{1}{2} \sum_m \sum_n B_{mn} \left(c_{ijkl} S_{ij}^{(m)} S_{kl}^{(n)} - 2e_{kij} E_k^{(m)} S_{ij}^{(n)} - \varepsilon_{kl} E_k^{(m)} E_l^{(n)} \right); \\
& S_{ij} = \sum_n S_{ij}^{(n)} x_2^n, E_i = \sum_n E_i^{(n)} x_2^n, i, j = 1,2,3; \\
& S_{ij}^{(n)} = \frac{1}{2}\left[u_{i,j}^{(n)} + u_{j,i}^{(n)} + (n+1)\left(\delta_{i2} u_j^{(n+1)} + \delta_{j2} u_i^{(n+1)} \right) \right], \\
& E_i^{(n)} = -\phi_{,i}^{(n)} - (n+1)\delta_{2i}\phi^{(n+1)}; \\
& T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + B_{nn} T_j^{(n)} = \rho \sum_m B_{mn} \ddot{u}_j^{(m)}, \\
& D_{i,i}^{(n)} - nD_2^{(n-1)} + B_{nn} D^{(n)} = 0; \\
& T_{ij}^{(n)} = \sum_m B_{mn} \left(c_{ijkl} S_{kl}^{(m)} - e_{kij} E_k^{(m)} \right), \\
& D_i^{(n)} = \sum_m B_{mn} \left(e_{ijk} S_{jk}^{(m)} + \varepsilon_{ij} E_j^{(m)} \right), \\
& T_j^{(n)} = \frac{b^n}{B_{nn}} \left[T_{2j}(b) - (-1)^n T_{2j}(-b) \right], \\
& D^{(n)} = \frac{b^n}{B_{nn}} \left[D_2(b) - (-1)^n D_2(-b) \right].
\end{aligned} \tag{1}
$$

Important variables in (1) include displacements $u_j\left(u_j^{(n)}\right)$, electrical potential $\phi\left(\phi^{(n)}\right)$, kinetic energy $\overline{K}$, strain energy $\overline{H}$, strains $S_{ij}\left(S_{ij}^{(n)}\right)$, stresses $T_{ij}\left(T_{ij}^{(n)}\right)$, electrical displacements $D_i\left(D_i^{(n)}\right)$, electrical fields $E_i\left(E_i^{(n)}\right)$, plate half thickness b, plate density ρ, and boundary terms $T_j^{(n)}$ and $D^{(n)}$. Material properties such as the elastic constants c_{ijkl}, piezoelectric constants e_{ijk}, and dielectric constants ε_{ij} are required for the constitutive relations.

In (1), the integration constant

$$
B_{mn} = \begin{cases} \dfrac{2b^{m+n+1}}{m+n+1}, & m+n = \text{even}, \\ 0, & m+n = \text{odd}. \end{cases} \tag{2}
$$

We also assume the thickness of platings on both sides of the plate is identical and denoted it as $2b'$, and the density of the platings as ρ'.

The tractions on the surfaces of the plated crystal are

$$
\begin{aligned}
T_{2j}(b) &= T'_{2j}(B) - 2b'\rho'\ddot{u}_j(b), \\
T_{2j}(-b) &= T'_{2j}(-B) + 2b'\rho'\ddot{u}_j(-b),
\end{aligned} \tag{3}
$$

where $B = b + 2b'$, and T'_{2j} are the surface tractions on the platings.

With (3), now we have

$$
\begin{aligned}
T_j^{(n)} &= \frac{1}{B_{nn}}\left[b^n T_{2j}(b) - (-b)^n T_{2j}(-b) \right] \\
&= \frac{b^n}{B_{nn}}\left[\left(T'_{2j}(B) - (-1)^n T'_{2j}(-B) \right) - 2b'\rho'\left(\ddot{u}_j(b) + (-1)^n \ddot{u}_j(-b) \right) \right] \\
&= \Gamma_j^{(n)} - \frac{b^n}{B_{nn}} 2b'\rho' \left[\ddot{u}_j(b) + (-1)^n \ddot{u}_j(-b) \right]
\end{aligned} \tag{4}
$$

For each vibration mode of long waves, we have

$$
u_j(b) = \sum_{n=0}^{\infty} b^n u_j^{(n)}, \quad u_j(-b) = \sum_{n=0}^{\infty} (-b)^n u_j^{(n)}, \tag{5}
$$

hence (4) will be rewritten as

$$
\begin{aligned}
B_{nn} T_j^{(n)} &= B_{nn}\Gamma_j^{(n)} - \sum_{m=0}^{\infty} 2b'\rho' 2b^{m+n} \ddot{u}_j^{(m)}, \\
&= B_{nn}\Gamma_j^{(n)} - \sum_{m=0}^{\infty} \rho (m+n+1) R B_{mn} \ddot{u}_j^{(m)}, \\
R &= \frac{2b'\rho'}{b\rho}.
\end{aligned} \tag{6}
$$

Now the equations of motion with the consideration of the mass effect of platings are

$$
T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + B_{nn} F_j^{(n)} = \rho \sum_m B_{mn} \left[1 + (m+n+1)R \right] \ddot{u}_j^{(m)}. \tag{7}
$$

The effect of electrodes is now reflected in the inertia terms in (7).

For simplification, some variables are normalized to the half thickness of plate through the considerations of

$$
u_j^{(n)} = \frac{U_j^{(n)}}{b^n}\left(\frac{x_1}{b}, \frac{x_3}{b} \right) e^{-i\omega t}, \ \omega_0 = \frac{\pi}{2b}\sqrt{\frac{c_{66}}{\rho}}, \ \omega = \omega_0 \Omega, \tag{8}
$$

and solutions will be in normalized frequency $\Omega = \omega / \omega_0$ and surface displacement will be $b^n u_j^{(n)}$. However, surface traction term will be $B_{nn} T_j^{(n)} / b^{n-1}$. It should be noted that the fundamental thickness-shear frequency ω_0 is an important parameter in the analysis with equations we present.

We now examine the Mindlin plate equations for the proper insertion of correction factors. By expanding 2D equations of motion (1) for straight-crested waves, we have

$$
\begin{aligned}
&\begin{Bmatrix} T_{1,1}^{(0)} \\ T_{6,1}^{(0)} \\ T_{5,1}^{(0)} \end{Bmatrix} = 2b\rho \begin{Bmatrix} \ddot{u}_1^{(0)} \\ \ddot{u}_2^{(0)} \\ \ddot{u}_3^{(0)} \end{Bmatrix} + \frac{2b^3}{3}\rho \begin{Bmatrix} \ddot{u}_1^{(2)} \\ \ddot{u}_2^{(2)} \\ \ddot{u}_3^{(2)} \end{Bmatrix}, \\
&\begin{Bmatrix} T_{1,1}^{(1)} \\ T_{6,1}^{(1)} \\ T_{5,1}^{(1)} \end{Bmatrix} - \begin{Bmatrix} T_{6}^{(0)} \\ T_{2}^{(0)} \\ T_{4}^{(0)} \end{Bmatrix} = \frac{2b^3}{3}\rho \begin{Bmatrix} \ddot{u}_1^{(1)} \\ \ddot{u}_2^{(1)} \\ \ddot{u}_3^{(1)} \end{Bmatrix} + \frac{2b^5}{5}\rho \begin{Bmatrix} \ddot{u}_1^{(3)} \\ \ddot{u}_2^{(3)} \\ \ddot{u}_3^{(3)} \end{Bmatrix}, \\
&\begin{Bmatrix} T_{1,1}^{(2)} \\ T_{6,1}^{(2)} \\ T_{5,1}^{(2)} \end{Bmatrix} - 2\begin{Bmatrix} T_{6}^{(1)} \\ T_{2}^{(1)} \\ T_{4}^{(1)} \end{Bmatrix} = \frac{2b^3}{3}\rho \begin{Bmatrix} \ddot{u}_1^{(0)} \\ \ddot{u}_2^{(0)} \\ \ddot{u}_3^{(0)} \end{Bmatrix} + \frac{2b^5}{5}\rho \begin{Bmatrix} \ddot{u}_1^{(2)} \\ \ddot{u}_2^{(2)} \\ \ddot{u}_3^{(2)} \end{Bmatrix}, \\
&\begin{Bmatrix} T_{1,1}^{(3)} \\ T_{6,1}^{(3)} \\ T_{5,1}^{(3)} \end{Bmatrix} - 3\begin{Bmatrix} T_{6}^{(2)} \\ T_{2}^{(2)} \\ T_{4}^{(2)} \end{Bmatrix} = \frac{2b^5}{5}\rho \begin{Bmatrix} \ddot{u}_1^{(1)} \\ \ddot{u}_2^{(1)} \\ \ddot{u}_3^{(1)} \end{Bmatrix} + \frac{2b^7}{7}\rho \begin{Bmatrix} \ddot{u}_1^{(3)} \\ \ddot{u}_2^{(3)} \\ \ddot{u}_3^{(3)} \end{Bmatrix}.
\end{aligned} \tag{9}
$$

For the stress-displacement relations, we have

$$
\begin{aligned}
T_p^{(n)} = \sum_{m=0}^{N} B_{mn} \Big\{ & c_{p1}u_{1,1}^{(m)} + c_{p2}(m+1)u_2^{(m+1)} + c_{p3}u_{3,3}^{(m)} + c_{p4}\left[u_{2,3}^{(m)} + (m+1)u_3^{(m+1)}\right] \\
& + c_{p5}\left(u_{1,3}^{(m)} + u_{3,1}^{(m)}\right) + c_{p6}\left[u_{2,1}^{(m)} + (m+1)u_1^{(m+1)}\right] \Big\}
\end{aligned} \tag{10}
$$

Now let us examine the thickness-shear and overtone modes by assuming

$$u_j^{(n)} = A_j^{(n)} e^{i\omega t},\ j = 1, n = 0,1,2,3. \tag{11}$$

Consequently, stress components related to these modes are

$$
\begin{aligned}
T_6^{(0)} &= 2bc_{66}u_1^{(1)} + 2b^3c_{66}u_1^{(3)}, \\
T_6^{(1)} &= \frac{4b^3}{3}c_{66}u_1^{(2)}, \\
T_6^{(2)} &= \frac{2b^3}{3}c_{66}u_1^{(1)} + \frac{6b^5}{5}c_{66}u_1^{(3)}.
\end{aligned} \tag{12}
$$

These modes can be separated into two groups as

$$
\begin{aligned}
-2bc_{66}u_1^{(1)} - 2b^3c_{66}u_1^{(3)} &= \frac{2b^3}{3}\rho\ddot{u}_1^{(1)} + \frac{2b^5}{5}\rho\ddot{u}_1^{(3)}, \\
-2b^3c_{66}u_1^{(1)} - \frac{18b^5}{5}c_{66}u_1^{(3)} &= \frac{2b^5}{5}\rho\ddot{u}_1^{(1)} + \frac{2b^7}{7}\rho\ddot{u}_1^{(3)},
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
2b\rho\ddot{u}_1^{(0)} + \frac{2b^3}{3}\rho\ddot{u}_1^{(2)} &= 0, \\
-\frac{8b^3}{3}c_{66}u_1^{(2)} &= \frac{2b^3}{3}\rho\ddot{u}_1^{(0)} + \frac{2b^5}{5}\rho\ddot{u}_1^{(2)}.
\end{aligned} \tag{14}
$$

with (11) and

$$\omega^2\rho = \frac{c_{66}}{b^2}\Omega^2, \tag{15}$$

we have

$$
\begin{aligned}
\left(1 - \frac{\Omega^2}{3}\right)A_1^{(1)} + b^2\left(1 - \frac{\Omega^2}{5}\right)A_1^{(3)} &= 0, \\
\left(1 - \frac{\Omega^2}{5}\right)A_1^{(1)} + b^2\left(\frac{9}{5} - \frac{\Omega^2}{7}\right)A_1^{(3)} &= 0,
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
\Omega^2 A_1^{(0)} + b^2\frac{\Omega^2}{3}A_1^{(2)} &= 0, \\
-\frac{\Omega^2}{3}A_1^{(0)} + b^2\left(\frac{4}{3} - \frac{\Omega^2}{5}\right)A_1^{(2)} &= 0.
\end{aligned} \tag{17}
$$

For the first- and third-order modes, we must have

$$\left(1 - \frac{\Omega^2}{3}\right)\left(\frac{9}{5} - \frac{\Omega^2}{7}\right) = \left(1 - \frac{\Omega^2}{5}\right)^2. \tag{18}$$

From (18), we can solve for the frequency of the coupled vibrations and it shows that the accuracy with these equations can be improved by introducing correction factors for higher-order modes also [9].

III. CORRECTION FACTORS FOR ELECTRODED PLATES

With higher-order Mindlin plate equations presented in previous section and the way the thin electrodes are considered for their mass effect, we can modify the equations with the inclusion of the electrode effect. This is important because the constant correction factors will not be able to make the results of electroded plates accurate.

A. *Natural Correction with the Consideration of Electrodes*

For natural correction, we write the stress equations (12) as

$$
\begin{aligned}
T_6^{(0)} &= 2bc_{66}\alpha^2u_1^{(1)} + 2b^3c_{66}\beta^2u_1^{(3)}, \\
T_6^{(1)} &= \frac{4b^3}{3}c_{66}\gamma^2u_1^{(2)}, \\
T_6^{(2)} &= \frac{2b^3}{3}c_{66}\alpha^2u_1^{(1)} + \frac{6b^5}{5}c_{66}\beta^2u_1^{(3)}.
\end{aligned} \tag{19}
$$

The correction factors are $\alpha = \kappa_6^{(0)}$, $\beta = \kappa_6^{(2)}$, and $\gamma = \kappa_6^{(1)}$. Since we do not consider the stiffness effect of electrodes, these correction is the same as for the unelectroded plates. However, the inertia terms will include the mass effect of the electrodes through the mass ratio R, which modify the frequency equation in (18) to

$$
\begin{aligned}
&\left[\alpha^2 - \frac{\Omega^2}{3}(1+3R)\right]\left[\frac{9}{5}\beta^2 - \frac{\Omega^2}{7}(1+7R)\right] \\
&= \left[\alpha^2 - \frac{\Omega^2}{5}(1+5R)\right]\left[\beta^2 - \frac{\Omega^2}{5}(1+5R)\right].
\end{aligned} \tag{20}
$$

Since exact frequencies of the thickness-shear modes are already known with the inclusion of electrodes [14, 15], we can use them to obtain correction factors to adjust the frequency solutions in (20) to the exact values. This is the procedure shown in [9] for unelectroded crystal plates. The numerical calculation of the correction factors from (20) with known frequency references is straight-forward.

B. Symmetric Correction with the Consideration of Electrodes

For a symmetric correction, we consider $\alpha = \kappa_6^{(0)}$ and $\beta = \kappa_6^{(2)}$, which result in stresses as [9]

$$\begin{aligned} T_6^{(0)} &= 2bc_{66}\alpha^2 u_1^{(1)} + 2b^3 c_{66}\alpha\beta u_1^{(3)}, \\ T_6^{(1)} &= \frac{4b^3}{3} c_{66}\gamma^2 u_1^{(2)}, \\ T_6^{(2)} &= \frac{2b^3}{3} c_{66}\alpha\beta u_1^{(1)} + \frac{6b^5}{5} c_{66}\beta^2 u_1^{(3)}. \end{aligned} \quad (21)$$

The resulted frequency equation with electrodes considered is

$$\left[\alpha^2 - \frac{\Omega^2}{3}(1+3R)\right]\left[\frac{9}{5}\beta^2 - \frac{\Omega^2}{7}(1+7R)\right] = \left[\alpha\beta - \frac{\Omega^2}{5}(1+5R)\right]^2, \quad (22)$$

which again requires the exact frequency solutions from an electroded plate to obtain the correction factors.

IV. Numerical Examples

For an electroded crystal plate with identical electrodes on both faces, we assume [14, 15]

$$\begin{aligned} u_1 &= A\sin\eta x_2, \quad \text{in crystal plate,} \\ \bar{u}_1 &= \pm A\sin\eta b\cos\bar{\eta}(x_2 \mp b) + \bar{B}\sin\bar{\eta}(x_2 \mp b), \quad \text{in electrodes.} \end{aligned} \quad (24)$$

With parameters

$$B = \frac{\bar{b}}{b}, C_{pq} = \frac{\bar{c}_{pq}}{c_{66}}, \xi = \eta b, \eta = k\bar{\eta}, \quad k^2 = \frac{\bar{v}_2^2}{v_2^2} = \frac{\dfrac{\bar{c}_{66}}{\bar{\rho}}}{\dfrac{c_{66}}{\rho}}, \quad (25)$$

the exact frequency of thickness-shear modes will be determined by [14]

$$\tan\xi \tan\frac{2B}{k}\xi = \frac{k}{C_{66}}. \quad (26)$$

Now we consider electrodes as gold with material properties

$\bar{c}_{66} = 2.71\times10^{10}\,\text{N/m}^2$, $\bar{\rho} = 18500\,\text{kg/m}^3$,

and the plate substrate is of AT-cut quartz crystal for our numerical calculation.

The procedure of the calculation will be the following: 1) Obtain the exact frequencies of thickness-shear modes from (26) with given material and structural parameters; 2) Use the two exact frequencies in (20) and (22) for equation sets with undetermined correction factors of natural and symmetric corrections. This is exactly the procedure by Mindlin for the correction of the first-order plate equations [2, 3, 5] and our approach for the higher-order equations [9]. These correction factors will refine the plate equations required for the analysis of high frequency vibrations in overtone modes, which are frequently encountered in high frequency resonators with precision requirements. This is particularly important in the systematic implementation of higher-order plate equations with the finite element method.

A. Natural Correction Factors with Mass Effect

With gold electrodes and AT-cut quartz crystal plate, we calculate the exact frequencies from (26) and the correction factors from (20). Now the results are plotted in Fig. 1. It is clear that the correction factor $\beta = \kappa_6^{(2)}$ is highly dependent on mass ratio R. Also it is interesting that the correction factors are more sensitive to mass ratio in the beginning. The effect of these correction factors is limited on the frequency spectra in the vicinity of the fundamental thickness-shear and the lower-order modes, but it has significant effect on certain modes as shown in the finite element analysis [11].

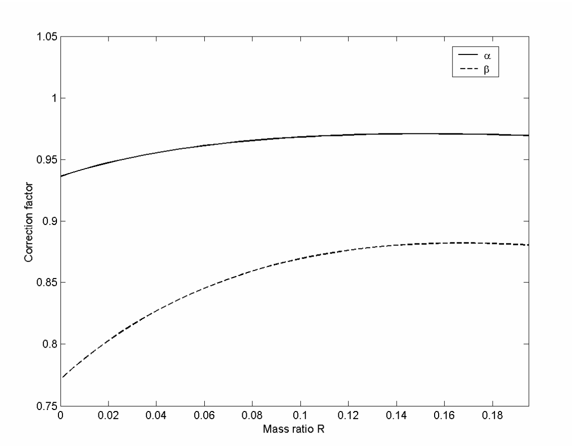

Fig. 1 Correction factors of the natural scheme for AT-cut quartz crystal plates with symmetric gold electrodes

B. Symmetric Correction Factors with Mass Effect

Similarly, we calculated the symmetric correction factors from (22) and plotted the results in Fig. 2. Since there are two sets of correction factors in this case, we select the one with smaller values for applications. Again, we see a strong dependence of $\beta = \kappa_6^{(2)}$ on the mass ratio R.

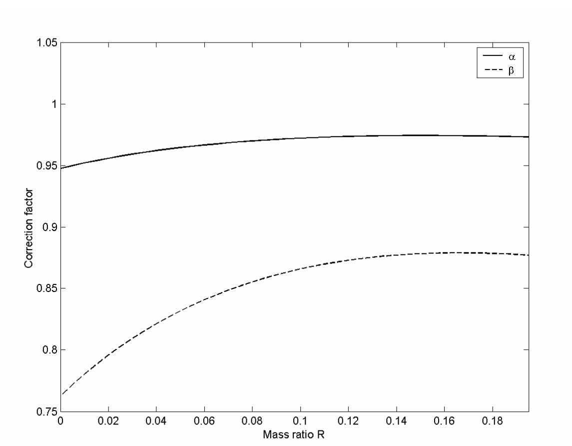

Fig. 2 Correction factors of the symmetric scheme for AT-cut quartz crystal plates with symmetric gold electrodes

V. Conclusions

In order to make Mindlin plate equations more accurate for high frequency vibrations involving higher-order overtone modes, we introduced correction factors in natural and symmetric manners to the higher-order equations. It has been shown that such corrections will make the overtone cut-off frequencies of the thickness-shear modes accurate, thus ensuring the applicability of these equations in a systematic manner. In this study, we obtained the correction factors with the consideration of mass effect of gold electrodes. The correcting procedure is exactly the one proposed and demonstrated by Mindlin, and multiple sets of correction factors are the results of enlarged number of equations.

These correction factors can also be calculated for other metal electrodes that are commonly used in resonators so the analysis can be more accurate. We have also obtained the correction factors as functions of the mass ratio so the calculations will be much simpler.

It should be pointed out that we restrict the applications of these correction factors to small mass ratios only because the higher-order equations we employed do not include the stiffness of electrodes, resulting inaccuracy in the thicker electrode cases. Since the exact frequencies have considered the stiffness of electrodes, there is an obvious discrepancy in the derivation. On the other hand, we have shown from earlier studies that the exact frequency is well approximated for thin electrodes with smaller mass ratios, which implies that our correction factors are also accurate for smaller mass ratios. Of course, correction factors can also be improved by including the stiffness of electrodes in the plate equations, which will result in a modified frequency equations also. Then, applicable range of correction factors can be extended to include thicker electrodes, which may appear more often in these days due to the continuous shrinking of resonators leading by utilizing smaller substrates. The direct implication is that electrodes are relatively enlarged, so is the effect of electrodes. In this case, the consideration of electrodes will be more important, and the stiffness will play the dominant role after the mass ratio has been included from the beginning. The applications of these correction factors will be in the analysis of higher-order overtone resonators and the finite element implementation of the Mindlin plate equations.

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